Euler and the Multiplication Formula for the Γ-Function.

Alexander Aycock

We show that the multiplication formula for the Γ-function was already found by Euler in [*E*421].

Contents

1 Introduction

In modern notation, the multiplication formula for the Γ-function reads as follows:

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{x-\frac{1}{2}}}\cdot\Gamma(x). \tag{1}$$

Here, n is a natural number and

$$\Gamma(x):=\int\limits_0^\infty e^{-t}t^{x-1}dt \quad \text{for} \quad \mathrm{Re}(x)>0.$$

(1) was first proven rigourously by Gauss in his influential paper [*Ga*28][1]. But we will show, how it can be derived from the results given by Euler in his lesser-known paper [*E*421], where he gave a formula equivalent to it.

2 Preparations

Of course, we need some formulas given by Euler in [*E*421] and some auxiliary formulas that can easily derived from Euler's results.

2.1 Expressing Euler's Formulas in modern Notation

In § 44 of his paper [*E*421], see page 343 in the Opera Omnia Version (he also studies these formulas extensively in [*E*321]), Euler defines

$$\left(\frac{p}{q}\right)=\int\limits_0^1\frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}}.$$

By the substitution $x^n=y$ it is easily seen that

$$\left(\frac{p}{q}\right)=\frac{1}{n}\int\limits_0^1 y^{\frac{p}{n}-1}dy(1-y)^{\frac{q}{n}-1}=\frac{1}{n}B\left(\frac{p}{n},\frac{q}{n}\right), \tag{2}$$

where

[1]Gauss was appearently not aware of Euler's work on the multiplication formula. For, he cited Euler's results very often but did not mention [*E*421] in any of his papers.

$$B(x,y) = \int_0^1 t^{x-1} dt (1-t)^{y-1} \quad \text{for} \quad \mathrm{Re}(x), \mathrm{Re}(y) > 0$$

is the Beta function. Euler implicitly assumes p and q to be natural numbers in $\left(\frac{p}{q}\right)$. This restriction is not necessary, of course.

2.2 Two auxiliary Formulas

First Formula

We have

$$\prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right) = \pi^{\frac{n-1}{2}} \sqrt{\prod_{i=1}^{n-1} \sin\left(\frac{i\pi}{n}\right)}. \tag{3}$$

To see this, consider the reflection formula for the Γ-function, also given by Euler in § 43 of [*E*421]. In the Opera Omnia Version the formula can be found on page 342. He writes it as $[\lambda] \cdot [-\lambda] = \frac{\lambda\pi}{\sin \pi\lambda}$ and $[\lambda]$ stands for $\lambda!$. In modern, notation we have

$$\frac{\pi}{\sin \pi x} = \Gamma(x)\Gamma(1-x).$$

Now, just apply this for $x = \frac{i}{n}$ with $i = 1, 2, \cdots n-1$. Then

$$\left.\begin{array}{rcl} \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right) & = & \frac{\pi}{\sin\frac{\pi}{n}} \\ \Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right) & = & \frac{\pi}{\sin\frac{2\pi}{n}} \\ \vdots & & \vdots \\ \Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right) & = & \frac{\pi}{\sin\frac{(n-1)\pi}{n}} \end{array}\right\} \quad \Rightarrow \quad \prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right)^2 = \pi^{\frac{n-1}{2}} \prod_{i=1}^{n-1} \sin\left(\frac{i\pi}{n}\right)$$

(3) now follows by taking the square root.

Second Formula

We have

$$\prod_{i=1}^{n-1} \sin\left(\frac{i\pi}{n}\right) = \frac{n}{2^{n-1}}. \tag{4}$$

This is an elementary formula. You can find a derivation, e.g., in [Fr06] (p. 17, ex. 19). The first and the second auxiliary formula were also proved by Gauss in [Ga28] and are crucial in his proof of the multiplication formula.

3 Derivation of the Product Formula

In § 53 of [E421], see page 348 in the Opera Omnia Version, Euler gives the formula

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)}.$$

Euler uses $[x]$ to denote the factorial of x so that in our notation $\left[\frac{m}{n}\right] = \Gamma\left(\frac{m}{n}+1\right)$. Euler assumes m and n to be natural numbers. But it is easily seen that we can interpolate $1\cdot 2\cdot 3\cdots(m-1)$ by $\Gamma(m)$. Therefore, let us assume x to be real and $x > 0$ and let us write x instead of m in the above formula. Further, using (2), Euler's formula reads

$$\Gamma\left(\frac{x}{n}\right) = \sqrt[n]{n^{n-x}\Gamma(x)\frac{1}{n^{n-1}}B\left(\frac{1}{n},\frac{x}{n}\right)B\left(\frac{2}{n},\frac{x}{n}\right)\cdots B\left(\frac{n-1}{n},\frac{x}{n}\right)}.$$

Now, we have the following relation among the B- and Γ- function, also given by Euler in the Supplement of [E421]. See page 354 in the Opera Omnia Version. The first to prove it rigorously was Jacobi in [Ja33]. Of course, we have

$$B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}.$$

Substituting the right-hand side for each B-function and after some simplification under the $\sqrt[n]{}$-sign we will find

$$\Gamma\left(\frac{x}{n}\right) = \sqrt[n]{n^{1-x}\Gamma(x)\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+1}{n}\right)} \cdot \frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+2}{n}\right)} \cdots \frac{\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+n-1}{n}\right)}}.$$

Now, let us simplify this by bringing all Γ-functions of fractional argument to the left-hand side. We will find

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = n^{1-x}\Gamma(x)\Gamma\left(\frac{1}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right).$$

The product on the right-hand side, $\Gamma\left(\frac{1}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$, can be simplified by means of equation (3) and (4) so that we arrive at

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = n^{1-x}\Gamma(x)\sqrt{\pi^{n-1}\frac{2^{n-1}}{n}} = (2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-x}\Gamma(x).$$

Thus, we arrived at the multiplication formula (1).

4 Conclusion

Now we saw that Euler already found the multiplication formula for the Γ-function. He just expressed it in terms of the symbol $\left(\frac{p}{q}\right)$ or, in modern notation, in terms of $B(p,q)$. He probably did not transform it, as we did here, since he wanted to express the factorial of rational numbers in terms of integrals of *algebraic* functions. Therefore, he probably also did not replace $1 \cdot 2 \cdot 3 \cdots (m-1)$ by $\Gamma(m)$.

Therefore, this paper of Euler should provide some motivation to go through other papers written by Euler (and other mathematicians, of course) carefully and try to find some more results he discovered, although he might expressed them differently and they were proven rigorously by his successors. This will certainly be of interest for anyone studying the history of mathematics.

References

[*E*321] L. Euler *Observationes circa integralia formularum* $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ *posito post integrationem* $x=1$, Melanges de philosophie et de la mathematique de la societe royale de Turin 3, 1766, pp. 156-177"', reprinted in "'*Opera Omnia*: Series 1, Volume 17, pp. 269 - 288

[$E421$] L. Euler *Evolutio formulae integralis* $\int x^{f-1}dx\,(\log(x))^{\frac{m}{n}}$ *integratione a valore* $x=0$ *ad* $x=1$ *extensa* , Novi Commentarii academiae scientiarum Petropolitanae 16, 1772, pp. 91-139 , reprint: Opera Omnia: Series 1, Volume 17, pp. 316 - 357

[$Fr06$] E. Freitag, R. Busam *Funktionentheorie 1*, Springer (4. korrigierte und erweiterte Ausgabe)

[$Ga28$] C. Gauss *Disquisitiones generales circa seriem infinitam* $1+\frac{a\cdot b}{1\cdot c}x+\frac{a(a+1)\cdot b(b+1)}{1\cdot 2\cdot c(c+1)}x^2+\cdots$, *Commentationes recentiores Bd. II. , Göttingen 1813*

[$Ja33$] C. Jacobi *Demonstratio Formulae* $\int\limits_0^1 w^{a-1}(1-w)^{b-1}dw = \frac{\int\limits_0^\infty e^{-x}x^{a-1}dx\int\limits_0^\infty e^{-x}x^{b-1}dx}{\int\limits_0^\infty e^{-x}x^{a+b-1}dx} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, Crelle Journal für reine und angewandte Mathematik, Band 11, p. 307, 1833